

Topological Aspects of First-Order Neural-Symbolic Integration

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Invited talk for the lecture on Logic Programming and Connectionist Systems
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INTERNATIONAL CENTER
FOR COMPUTATIONAL LOGIC



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Hölldobler, Kalinke, Störr 1999 [HKS99]

First-order logic programs (acyclic, with injective level mapping) can be approximated by standard sigmoidal feedforward networks.

Existential result via continuity of the T_P -operator.

Features of the approach:

- Shows representability *in principle*.
- Single existing approach for first-order LPs with function symbols.

Limitations of the approach:

- No algorithm how to construct the network.
- Very restricted class of programs.
- Proofs highly dependent on specific assumptions.

Today: Lifting to a more general setting.

Next week: How to construct the networks.

Embedding T_P into the reals

$T_P : I_P \rightarrow I_P$, where $I_P = 2^{B_P} \approx 2^{\mathbb{N}}$

$2^{\mathbb{N}}$: all (countably infinite) binary sequences

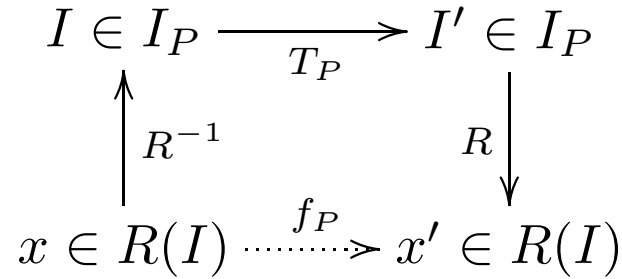
\rightsquigarrow interpret sequence as expansion in b -adic number system! ($b > 2$)

1. Choose an *enumeration* of B_P (= bijective level mapping).
2. Choose a base $b > 2$ for the number system.
3. Choose integers $t \neq f$ with $0 \leq t, f < b$.
4. Given an interpretation $I \subseteq B_P$, set $a_i = f$ if the i -th element of I is false, and set $a_i = t$ otherwise.
5. Set $R(I) = \sum_{i=0}^{\infty} a_i b^i$.

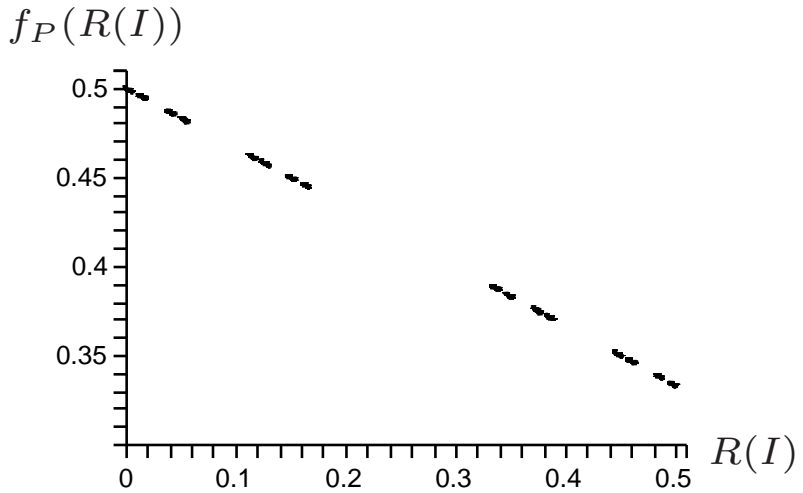
Denote $R(I_P)$ by D_R .

Embedding T_P into the reals

Set $f_P : D_R \rightarrow D_R : x \mapsto R(T_P(R^{-1}(x)))$.



$\text{even}(0)$.
 $\text{even}(s(X)) \leftarrow \neg \text{even}(X)$.



- Domain of f_P is *totally disconnected*. (Is *Cantor set*: discussed later.)
 \rightsquigarrow There is a *gap* between any two points.

Approaching Topology

- [Funahashi 89] treats *continuous functions* (on \mathbb{R}^n).
- We have seen *metrics* (distance functions) playing a role both on \mathbb{R} and on I_P .
- Iterative behaviour of T_P (*convergence* of iterates) important.
 - Continuity, metrics, and convergence are notions known from \mathbb{R} .
 - They can be studied on much more general spaces (including I_P).
 - Corresponding mathematical subject: (Set-theoretic) *Topology*.

Topology:

- Qualitative and quantitative (metric) notions of *distance*.
- Continuity of functions in the sense of *preservation of limits*.
- Notions of approximation in various spaces.
- Abstracts, e.g. allows to link \mathbb{R} and I_P in a sound way.
- Bridge between the continuous (networks) and the discrete (logics).

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Standard (natural) topology on \mathbb{R}

Open intervals: known.

Open sets (opens): All unions of open intervals.

Open neighborhood of $x \in \mathbb{R}$: Open set containing x .

The following hold:

- (i) All unions of opens are open.
- (ii) All *finite* intersections of opens are open.

Conditions (i) and (ii) define: The set of all opens is a *topology* on \mathbb{R} .

Note: Infinite intersections of open intervals are not necessarily open.

$$\text{E.g. } \bigcap_{n=1}^{\infty} \left] 0 - \frac{1}{n}, 1 + \frac{1}{n} \right[= [0, 1].$$

\rightsquigarrow A **topology** \mathcal{O} on a set X is a set of subsets of X s.t. (i) and (ii) hold.

Note: The *empty* union (= all of X) is in every topology!

Standard (natural) topology on \mathbb{R}

Standing assumption (for us): There exists a countable subset \mathcal{B} of \mathcal{O} s.t. every open is a union of members of \mathcal{B} .

\mathcal{B} called a *base* for \mathcal{O} .

Condition called: \mathcal{O} is *second countable* (C2).

For \mathbb{R} : Take as base e.g. all intervals with rational endpoints.

\mathcal{S} is a *subbase* of \mathcal{O} if the set of all finite intersections of sets in \mathcal{S} are a base for \mathcal{O} .

Subspace: A subset $X \subseteq \mathbb{R}$ inherits a topology from \mathbb{R} , consisting of all $O \cap X$ for all $O \in \mathcal{O}$.

Example: $[0, 1]$ as subspace of \mathbb{R} .

Some open sets of $[0, 1]$: \emptyset $[0, 1]$ $[0, 0.5[$ $]0.1, 0.6[\dots$

Standard (natural) topology on \mathbb{R}

Convergence: Sequence (x_n) converges to x (written $x_n \rightarrow x$) iff (x_n) is *eventually* in every open neighborhood of x .

Note: Limits are not necessarily unique.

Example: $X = \{0, 1\}$, $\mathcal{O} = \{\emptyset, X\}$.

Then the sequence $1, 1, 1, \dots$ converges to 0 and 1.

Continuity: A function $f : \mathbb{R} \rightarrow \mathbb{R}$ is *continuous* iff the pre-image $f^{-1}(O)$ is open for each open O .

Then: $x_n \rightarrow x$ implies $f(x_n) \rightarrow f(x)$.

This condition *characterizes* continuity on many spaces (e.g. C^2 spaces).

All definitions hold also for more general topologies.

Continuity: Example

Functions on \mathbb{R} are continuous if they *can be drawn without lifting the pen*.

$$x \mapsto 2x \quad x \mapsto x^2 \quad x \mapsto \ln x \quad x \mapsto 5 \sin \frac{x}{\pi} \dots$$

Now consider the subspace $X = [1, 2] \cup [3, 4] \subseteq \mathbb{R}$.

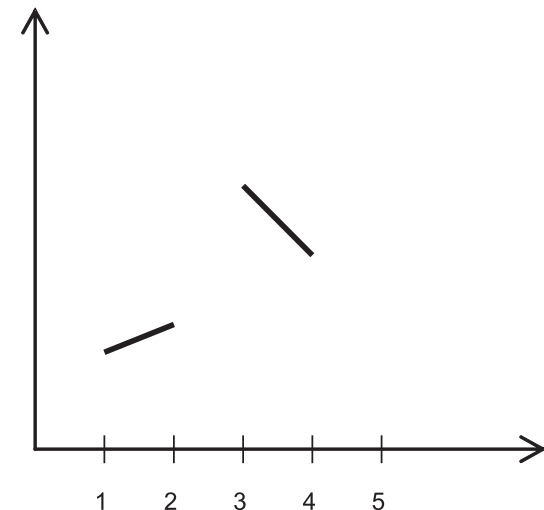
X consists of two connected pieces.

X is *not connected* (as a whole).

$$f : X \rightarrow \mathbb{R}$$

is continuous iff *on each of the connected parts it can be drawn without lifting the pen*.

See example on the right.



Cantor Space

A subspace of \mathbb{R} .

$$C = [0, 1] \cap \left(\left[0, \frac{1}{3} \right] \cup \left[\frac{2}{3}, 1 \right] \right) \cap \left(\left[0, \frac{1}{9} \right] \cup \left[\frac{2}{9}, \frac{1}{3} \right] \cup \left[\frac{2}{3}, \frac{7}{9} \right] \cup \left[\frac{8}{9}, 1 \right] \right) \cap \dots$$



- Encoding 0=left, 1=right, each point $x \in C$ is an (infinite) binary sequence $\text{seq}(x)$.
- C is uncountable.
- There is a *gap* between each two points of C .
- A base: $\{B_n(x) \mid x \in C\}$, where
 $B_n(x) = \{y \in C \mid \text{seq}(x), \text{seq}(y) \text{ coincide on the first } n \text{ digits}\}.$

Metrics

A *metric* on a set X is a function $d : X \times X \rightarrow \mathbb{R}$ which satisfies the following for all $x, y, z \in X$.

- (i) $d(x, y) = 0$ iff $x = y$.
- (ii) $d(x, y) = d(y, x)$.
- (iii) $d(x, z) \leq d(x, y) + d(y, z)$. (triangle inequality)

- Abstract notion of *distance*.

E.g. on \mathbb{R} : $d(x, y) = |x - y|$.

Collection of all $B_\varepsilon(x) = \{y \in X \mid d(x, y) < \varepsilon\}$ is base of a topology.

But not every topology comes from a metric!

- $B_\varepsilon(x)$ is called the *open ball with center x and radius ε* .
- Every metric induces exactly one topology!

Metrics on Cantor Space

Metric inherited from \mathbb{R} :

For $x, y \in C$ set $d(x, y) = |x - y|$.

Induced topology: subspace topology inherited from \mathbb{R} .

Prefix distance on sequences:

For $x, y \in C$ set $\delta(x, y) = 2^{-n}$,

where n is least s.t. $\text{seq}(x)$ and $\text{seq}(y)$ differ on the n -th digit.

Prefix distance produces base $\{B_{2^{-n}}(x) \mid x \in C\}$, where

$B_{2^{-n}}(x) = \{y \in C \mid \text{seq}(x), \text{seq}(y) \text{ coincide on the first } n \text{ digits}\}$.

\rightsquigarrow The base mentioned before!

We will call the prefix distance also *Fitting metric*.

Cantor space

- Can be characterized by the following properties: totally disconnected, compact, Hausdorff, second countable, dense in itself.
- It is a very *well-known* and omnipresent topological space.

- Cantor space sometimes known as *Cantor dust*.
- Cantor space is *self-similar*.
 Zooming in on it produces a very similar picture.
- It is a *fractal* as known from *Chaos theory* or *topological dynamics*.
- Can be “produced” by iterated function systems.
 \rightsquigarrow More about this in two weeks!

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A topology on I_P : The atomic topology Q

[Batarekh & Subrahmanian 1989] [Seda 1995]

For each finite conjunction $C = L_1 \wedge \cdots \wedge L_n$ of literals,

set $\mathcal{G}(C) = \{I \in I_P \mid I \models C\}$.

$\{\mathcal{G}(C) \mid C \text{ a finite conjunction of literals}\}$ is a *base* of the topology Q .

A sequence (I_n) in I_P converges wrt. Q to some $I \in I_P$ iff

- For each $A \in I$ exists n_0 s.t. for all $k \geq n_0$: $A \in I_k$.
- For each $A \notin I$ exists n_1 s.t. for all $m \geq n_1$: $A \notin I_m$.

A result:

If $T_P^n(K) \rightarrow I$ then $I \subseteq T_P(I)$, i.e. I is a *model* of P .

If T_P is also continuous, then $I = T_P(I)$.

A topology on I_P : The atomic topology Q

Another base for Q :

- Fix an enumeration (injective level mapping) on B_P : A_1, A_2, A_3, \dots
- Let C be conjunctions of the form L_1, \dots, L_n , where L_i is A_i or $\neg A_i$.

Set $\mathcal{G}(C) = \{I \in I_P \mid I \models C\}$ as before.

Then $\{\mathcal{G}(C) \mid C \text{ of the indicated form}\}$ is a base of the topology Q .

(Clear: This is a subbase. But it really is also a base!)

Metric characterization of Q :

- $d(I, K) = 2^{-n}$, where n is least such that the enumerations of elements in I and K differ at the n -th position.

\rightsquigarrow Exactly the same as the prefix distance (Fitting metric)!

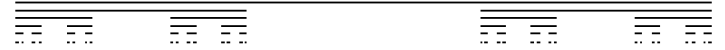
I_P and \mathbb{R} — topological link

Cantor space:

binary sequence

(via the graphical construction)

$\rightsquigarrow x \in C \subseteq \mathbb{R}$



I_P with Q :

binary sequence (via enumeration of interpretation)

$\rightsquigarrow I \in I_P$

Fitting metric (prefix distance) topologizes both spaces.

$\Rightarrow I_P$ with Q is topologically *the same* as C !

$\Rightarrow f_P$ is continuous *iff* T_P is!

$$\begin{array}{ccc}
 I \in I_P & \xrightarrow{T_P} & I' \in I_P \\
 \uparrow \text{seq} & & \text{seq}^{-1} \downarrow \\
 x \in C & \xrightarrow{\dots f_P \dots} & x' \in C
 \end{array}$$

[HKS99] in topological light

[Hitzler, Hölldobler & Seda, 2004]

- Injective level mapping = enumeration of B_P .
 - $R : I_P \rightarrow C$ has Cantor space as range.
(may be different concrete set on \mathbb{R} , but topologically the same)
 - R is a *topological homeomorphism*
(mapping preserving the topological structure).
 - Acyclicity of the program: guarantees continuity of T_P .
 \leadsto More about this later.
- \Rightarrow Hence: f_P as representation of T_P on \mathbb{R} is continuous and can be approximated by feedforward networks!

$$\begin{array}{ccc} I \in I_P & \xrightarrow{T_P} & I' \in I_P \\ \uparrow R^{-1} & & \downarrow R \\ x \in R(I) & \xrightarrow{f_P} & x' \in R(I) \end{array}$$

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Funahashi's theorem revisited

Theorem (Funahashi 1989, simplified version):

σ sigmoidal

$K \subseteq \mathbb{R}$ compact,

$f : K \rightarrow \mathbb{R}$ continuous,

$\varepsilon > 0$.

Then there exists perceptron with sigmoidal σ and I/O-function $\bar{f} : K \rightarrow \mathbb{R}$ with

$$\max_{x \in K} \{d(f(x), \bar{f}(x))\} < \varepsilon;$$

d metric which induces natural topology on \mathbb{R} .

I.e. every continuous function $f : K \rightarrow \mathbb{R}$ can be uniformly approximated by I/O-functions of perceptrons.

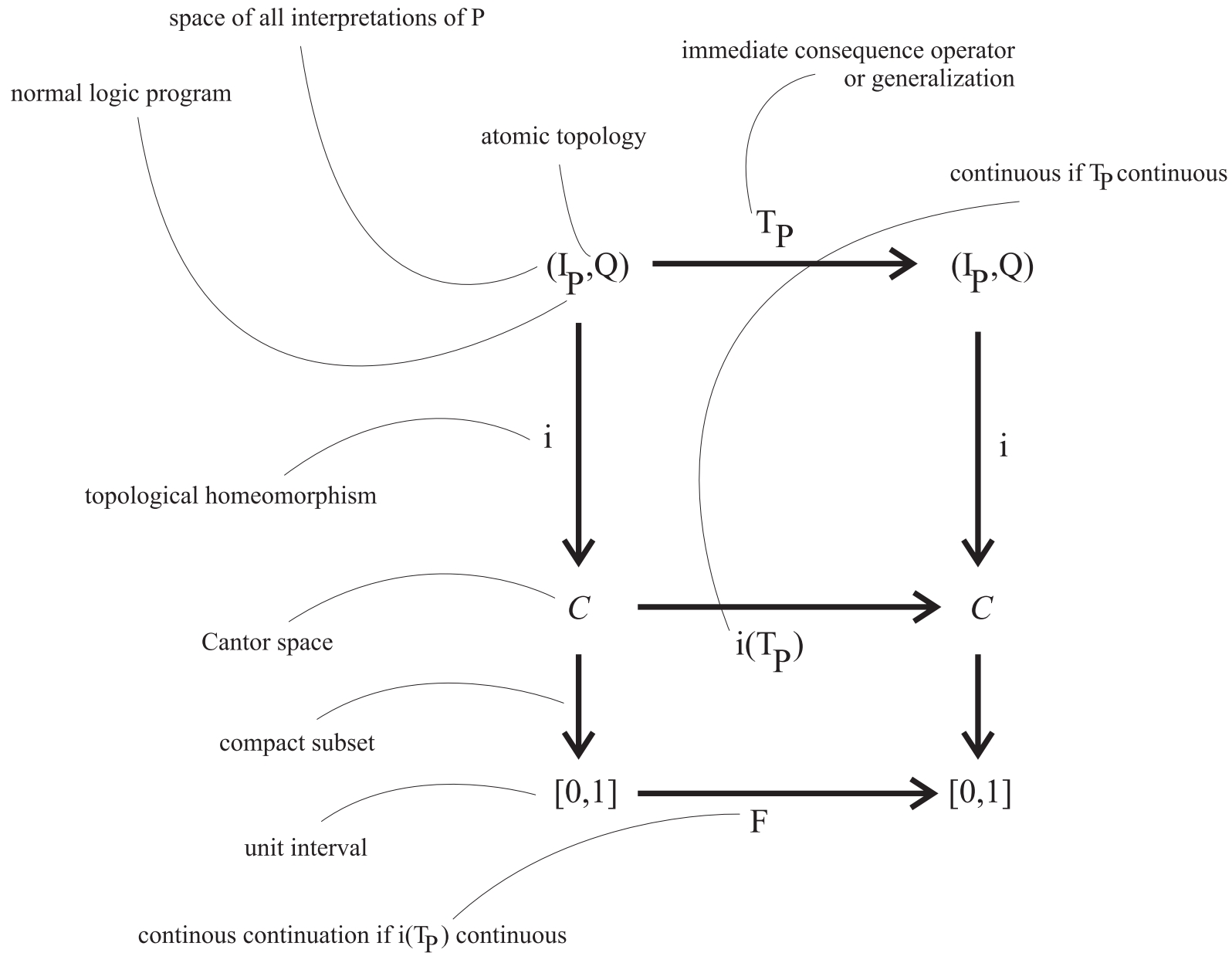


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General consequence operators: multi-valued logics

- Truth values $\mathcal{T} = \{t_1, \dots, t_n\}$.
- Interpretations are functions $I : B_P \rightarrow \mathcal{T}$.
- $I_{P,n} = I_P$ set of all interpretations.
- \mathcal{B}_A set of all atoms in bodies of clauses in $\text{ground}(P)$ with head $A \in B_P$.
- $T : I_P \rightarrow I_P$ consequence operator for P , if for all $I \in I_P$ and all $A \leftarrow \text{body}$ in P we have that $T(I)(A) \leftarrow I(\text{body})$ holds via truth table.
- T local if $T(I)(A) = T(K)(A)$ for all $A \in B_P$ and all $I, K \in I_P$ which agree on \mathcal{B}_A .
- T_P is a local consequence operator.

Other examples: Operators as defined by Fitting (1985,199x) in three- or four-valued logic.

Cantor topology \mathcal{Q}

- For \mathcal{Q} , we considered *binary* sequences (two truth values).
- Now consider sequences, where each element can be one out of n values (n truth values).

Use the same prefix distance:

For $I, K \in I_{P,n}$ set $\delta(I, K) = 2^{-n}$,

where n is least s.t. I and K differ on the n -th element.

Topological structure turns out to be the same!

$\Rightarrow \mathcal{Q}$ is *the same as* the Cantor topology!

$\Rightarrow I_{P,n}$ is *the same as* Cantor space!

Approximation of continuous consequence operators

Theorem (Hitzler & Seda 2003)

Let P be a logic program, T be a locally finite consequence operator, and ι be a homeomorphism from $(I_{P,n}, \mathcal{Q})$ to \mathcal{C} .

Then $\iota(T)$ can be uniformly approximated by I/O-functions of 3lfn.

This holds *mutatis mutandis* e.g. for radial basis function networks (activation function is gaussian).

ι normally given via some enumeration (injective level mapping) $l : B_P \rightarrow \mathbb{N}$ and some corresponding p -adic expansion.

There exist *uncountably many homeomorphisms* from I_P to \mathcal{C} .

Lots of degrees of freedom!

Characterizing continuity in \mathcal{Q}

Consequence operator T on I_P is *locally finite*, if for all $A \in \mathcal{B}_P$ and all $I \in I_P$ there exists a finite set $S \subseteq \mathcal{B}_A$ with $T(J)(A) = T(I)(A)$ for all $J \in I_P$ which agree with I on S .

Theorem

A local consequence op. is locally finite iff it is continuous in \mathcal{Q} .

Sufficient:

- P is *covered*, i.e. does not contain any *local variables* (occurring in some body, but not in corresponding head).

Continuity of T_P

Continuity of T_P is guaranteed under any of the following conditions:

- P is propositional.
- P does not contain function symbols.
- P is acyclic with respect to an injective level mapping.
- P is covered.

Open Question

Describe a maximal class \mathcal{A} of programs such that for each $P \in \mathcal{A}$ there exists a *covered* program Q with $T_Q = T_P$.

Does \mathcal{A} contain *all* programs P with continuous T_P ?

Probably not, but it may contain all

computationally relevant such programs.

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Recursive architecture: Reasoning

T locally finite consequence operator. f approximating I/O-function.

- For all $I \in I_P$ and $n \in \mathbb{N}$ we have $|f^n(\iota(I)) - \iota(T^n(I))| \leq \varepsilon \frac{1-\lambda^n}{1-\lambda}$.

Need: λ Lipschitz-constant of F (i.e. $|F(x) - F(y)| \leq \lambda|x - y|$ for all x, y), and F the continuation of $\iota(T)$ to $[0, 1]$.

ε bound on approximation error.

- If F is contraction, then $(F^k(\iota(I)))$ converges for all I to unique fixed point x of F and $\exists m \in \mathbb{N} \forall n \geq m: |f^n(\iota(I)) - x| \leq \varepsilon \frac{1}{1-\lambda}$.
Furthermore, T is a contraction on the complete space I_P (with suitable metric), and we have $\iota(M) = x$ for the unique fixed point M of T .

- Assume there is $I \in I_P$ s.t. $T^n(I)$ converges in \mathcal{Q} to a fixed point M of T .

Then for every $\delta > 0$ there exists some $n \in \mathbb{N}$ and a network with $|f^n(\iota(I)) - \iota(M)| < \delta$.

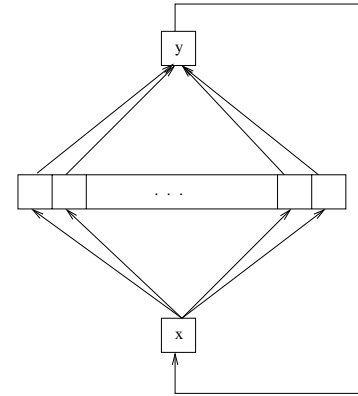


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Remarks on measurability

An alternative to [Funahashi 89]:

Theorem (Hornik, Stinchcombe, White 1989, simplified version)

$\sigma : \mathbb{R} \rightarrow (0, 1)$ monotonic increasing, onto.

$f : \mathbb{R} \rightarrow \mathbb{R}$ Borel measurable,

μ Borel probability measure on \mathbb{R} ,

$\varepsilon > 0$.

Then there is a perceptron with sigmoidal activation function σ and I/O-function $\bar{f} : \mathbb{R} \rightarrow \mathbb{R}$ with

$$\varrho_{\mu}(f, \bar{f}) = \inf \{ \delta > 0 : \mu \{ x : |f(x) - \bar{f}(x)| > \delta \} < \delta \} < \varepsilon.$$

I.e. the set of I/O-functions which can be computed using 3lfns is dense with respect to ϱ_{μ} in the set of all Borel measurable functions $f : \mathbb{R} \rightarrow \mathbb{R}$.

Measurable consequence operators

Theorem (Hitzler & Seda 2000)

Local consequence operators are always measurable with respect to $\sigma(Q)$.

But:

Approximation by networks is only *almost everywhere*.

Cantor set has measure 0.

Result can be improved to some extent, but principal problem remains.

⇒ Continuity approach appears to be more promising!

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Non-monotonic reasoning

Fixed points of operator GL_P yield the *stable models* of P .
(as in Answer Set Programming)

[DK89] Phan M. **Dung** and Kanchana **Kanchanasut**, A fixpoint approach to declarative semantics of logic programs. Proc. NACLP'89, 1989.

Program transformation $P \mapsto \text{fix}(P)$.

Complete unfolding through positive body literals.

[Wen02] Matthias **Wendt**, Unfolding the well-founded semantics, Journal of Electrical Engineering 2002.

Shows $GL_P(I) = T_{\text{fix}(P)}(I)$ for all interpretations I .

↪ Allows to carry over results. (Bader & Hitzler, in preparation)

↪ Works similarly also for well-founded semantics.

The Fixpoint Completion

Quasi-interpretation K : set of clauses of form $A \leftarrow \neg B_1, \dots, \neg B_m$.

Program P : set of (ground) clauses of form

$$A \leftarrow A_1, \dots, A_n, \neg B_1, \dots, \neg B_m.$$

$T'_P(K)$ set of $A \leftarrow \text{body}_1, \dots, \text{body}_n, \neg B_1, \dots, \neg B_m$

where $A \leftarrow A_1, \dots, A_n, \neg B_1, \dots, \neg B_m$ in P

and $A_i \leftarrow \text{body}_i$ in K for all i .

$T'_P \uparrow \omega = \text{fix}(T'_P) = \text{fix}(P)$ quasi-interpretation.

Thank You!

Pascal Hitzler



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